Algorithm Derivation of Orthotropic Material Simulation by Non-Linear FEM

Yan Wenyuan

May 11, 2016

1 Introduction

Some text goes here.

2 Algorithm Derivation

Similar to the transversely isotropic materials strain energy , we can split the orthotropic material strain energy into

$$\Psi = \Psi_{iso} + \Psi_{ortho} \tag{1}$$

$$\Psi_{ortho} = w_1(E_1; \bar{\lambda_1}) + w_2(E_2; \bar{\lambda_2}) + w_3(E_3; \bar{\lambda_3}) \tag{2}$$

Ψ_{iso} is isotropic part energy, and Ψ_{ortho} is orthotropic part energy; $w_1(E_1; \bar{\lambda_1})$, $w_2(E_2; \bar{\lambda_2})$, $w_3(E_3; \bar{\lambda_3})$ are all 1D energy function, to account for the energy of the three orthotropic material axes; $\bar{\lambda_i}$ is the strain along the three orthotropic material axes, and $\bar{\lambda_i} = ||Fm_i||_2$.

For orthotropic material, we also can compute the tangent stiffness matrix following Irving’ s work [Irving et al. 2004]. The gradient of the elastic force of a tetrahedron element is:

$$\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{G}}{\partial \boldsymbol{F}} \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}} = (\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}} \boldsymbol{B_m}) \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}, \tag{3}$$

where $x \in \boldsymbol{R}^{12}$, is the displacement of the four vertices. $F = Ds(\boldsymbol{x})Dm^{-1}$, so $\partial \boldsymbol{F}/\partial \boldsymbol{x}$ is constant, and $\boldsymbol{B_m}$ is also constant during the simulation. So, we need to compute $\partial \boldsymbol{P}/\partial \boldsymbol{F}$ in each time step.

Since $\mathbf{P}$ is defined using SVD, we can compute $\partial \boldsymbol{P}/\partial \boldsymbol{F}$ using product rule as follows

$$\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}_{ij}} = \frac{\partial \boldsymbol{U}}{\partial \boldsymbol{F}_{ij}} P(\hat{\boldsymbol{F}}) V^T + U \frac{\partial \boldsymbol{P}(\hat{\boldsymbol{F}})}{\partial \boldsymbol{F}_{ij}} V^T + U P(\hat{\boldsymbol{F}}) \frac{\partial \boldsymbol{V^T}}{\partial \boldsymbol{F}_{ij}} \tag{4}$$

The section 2.1, section 2.2 and section 2.3 will derivate the expression of $P(\hat{\boldsymbol{F}})$, $\boldsymbol{\partial P(\hat{F})/\partial F}_{ij}$ $\partial \boldsymbol{U}/\boldsymbol{F}_{ij}$ and $\partial \boldsymbol{V^T}/\partial \boldsymbol{F}_{ij}$.

2.1 $P(\hat{\boldsymbol{F}})$

For orthotropic material, $P(\hat{\boldsymbol{F}})$ can be computed as follows

$$P(\hat{\boldsymbol{F}}) = \frac{\boldsymbol{\partial \Psi(\hat{F})}}{\boldsymbol{\partial \hat{F}}} = \frac{\boldsymbol{\partial \Psi_{iso}(\hat{F})}}{\boldsymbol{\partial \hat{F}}} + \frac{\boldsymbol{\partial \Psi_{ortho}(\hat{F})}}{\boldsymbol{\partial \hat{F}}} = P_{iso}(\hat{\boldsymbol{F}}) + P_{ortho}(\hat{\boldsymbol{F}}), \tag{5}$$

and we put the equation(2) in (5), we can get the $P_{ortho}(\hat{\boldsymbol{F}})$ expression,

$$P_{ortho}(\hat{\boldsymbol{F}}) = \sum \frac{w_i'(\bar{\lambda}_i)}{\bar{\lambda}_i} \hat{\boldsymbol{F}}(\boldsymbol{V^T m_i}) \otimes (\boldsymbol{V^T m_i}) \tag{6}$$

where $\otimes$ is tensor product operation, $x \otimes y = xy^T$. Similarly, for orthotropic material, $\boldsymbol{\partial P(\hat{F})/\partial F}_{ij}$ can be computed as follows

$$\frac{\boldsymbol{\partial P(\hat{F})}}{\boldsymbol{\partial F}_{ij}} = \frac{\boldsymbol{\partial P_{iso}(\hat{F})}}{\boldsymbol{\partial F}_{ij}} + \frac{\boldsymbol{\partial P_{ortho}(\hat{F})}}{\boldsymbol{\partial F}_{ij}} \tag{7}$$

$$\begin{aligned} \frac{\boldsymbol{\partial P_{ortho}(\hat{F})}}{\boldsymbol{\partial F}_{ij}} &= \sum\Big(\frac{w_i'(\bar{\lambda}_i)}{\bar{\lambda}_i} \frac{\partial \hat{\boldsymbol{F}}}{\partial \boldsymbol{F}_{ij}} (\boldsymbol{V^T m_i}) \otimes (\boldsymbol{V^T m_i}) \\ &+ \hat{\boldsymbol{F}} \frac{w_i'(\bar{\lambda}_i)\bar{\lambda}_i - w_i'(\bar{\lambda}_i)}{\bar{\lambda}_i^{\,2}} \frac{\partial \bar{\lambda}_i}{\partial \boldsymbol{F}_{ij}} (\boldsymbol{V^T m_i}) \otimes (\boldsymbol{V^T m_i}) \\ &+ \frac{w_i'(\bar{\lambda}_i)}{\bar{\lambda}_i} \hat{\boldsymbol{F}}\big((\frac{\boldsymbol{V^T}}{\boldsymbol{F}_{ij}} \boldsymbol{m_i}) \otimes (\boldsymbol{V^T m_i}) + (\boldsymbol{V^T m_i}) \otimes (\frac{\boldsymbol{V^T}}{\boldsymbol{F}_{ij}} \boldsymbol{m_i})\big) \end{aligned} \tag{8}$$

$$\frac{\partial \bar{\lambda}_i}{\partial \boldsymbol{F}_{ij}} = \frac{1}{2\bar{\lambda}_i} \boldsymbol{m_i}^T \frac{\boldsymbol{\partial F^T F}}{\boldsymbol{\partial F_{ij}}} \boldsymbol{m_i}, \qquad \frac{\boldsymbol{\partial F^T F}}{\boldsymbol{\partial F_{ij}}} = F^{i,j} + (F^{i,j})^T \tag{9}$$

where $F^{i,j}$ is a matrix with all the elements 0, expect the (i, j) element equals $\boldsymbol{F}_{ij}$, and $\partial \hat{\boldsymbol{F}}/\partial \boldsymbol{F}_{ij}$ is a diagonal matrix, and proved

$$\text{diag}(\partial \hat{\boldsymbol{F}}/\partial \boldsymbol{F}_{ij}) = \text{diag}(U^T(\partial \boldsymbol{F}/\partial \boldsymbol{F}_{ij})V) \tag{10}$$

2.2 $\boldsymbol{\partial P(\hat{F})/\partial F}_{ij}$

Some text goes here.

2.3 $\partial \boldsymbol{U}/\boldsymbol{F}_{ij}$ and $\partial \boldsymbol{V^T}/\partial \boldsymbol{F}_{ij}$

Some text goes here.

3 Key Points

3.1 1D energy function w_i

Some text goes here.

3.2 The isotropic term energy Ψ_{iso}

Some text goes here.